

A GAUSSIAN UPPER BOUND OF THE CONJUGATE HEAT EQUATION ALONG AN EXTENDED RICCI FLOW

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ABSTRACT. In this paper, we derive a Sobolev inequality along an extended Ricci flow and prove a point-wise Gaussian type bound for the fundamental solutions of the conjugate heat equation under the flow.

1. INTRODUCTION

Let M^n be a n dimensional closed smooth manifold and assume $n \geq 3$. In [8], B. List studied a Ricci flow, coupled with a harmonic flow

$$(1.1) \quad \begin{cases} \partial_t g(x, t) = -2\text{Ric}(g(x, t)) + 4d\phi(x, t) \otimes d\phi(x, t), \\ \partial_t \phi(x, t) = \Delta_{g(x, t)} \phi, \end{cases}$$

where $g(x, t)$ is a family of Riemannian metrics, and $\phi(x, t)$ is a function on M for any fixed t . This flow is also called Ricci-Harmonic (RH) flow (c.f.[8, 15, 16, 20]). If ϕ is a constant function, the system (1.1) degenerates to Hamilton's Ricci flow discussed widely recently, see for example the book [2] and celebrated papers [4, 5, 6, 18, 19, 9]. The Ricci-Harmonic flow is very useful in general relativity. The stationary solutions (only depending on spatial variable x) of (1.1) are solutions to the following static Einstein vacuum system

$$\begin{cases} \text{Ric} = 2d\phi \otimes d\phi, \\ \Delta\phi = 0. \end{cases}$$

Similarly as Ricci flow, corresponding theories for the extended Ricci flow system was established in [8], such as the short time existence and the bounds of curvature tensor Rm and ϕ .

Let g be a Riemannian metric and ϕ be a smooth function on M . For the sake of conveniences, we denote as in [8] the symmetric tensor field $Sy \in \text{Sym}_2(M)$ and its trace $S := g^{ij} S_{ij}$ by

$$S_{ij} := R_{ij} - 2\partial_i \phi \partial_j \phi \quad \text{and} \quad S := R - 2|d\phi|^2.$$

Where R denotes the scale curvature of (M, g) . The summation convention of summing over repeated induces is used here and throughout the paper. Then the RH flow can be written as

$$\begin{cases} \partial_t g = -2Sy, \\ \partial_t \phi = \Delta\phi. \end{cases}$$

It is well-known that the Sobolev inequality contains a host of analytical and geometric information (c.f.[7, 11, 14]) including non-collapsing, isoperimetric inequalities and so on. It is also an important tool in studying elliptic and parabolic differential equations on manifolds (c.f.[11]). In [10, 21, 22, 23], via the monotonicity of Perelman's W entropy, some uniform Sobolev inequalities were proved along Ricci flow. As a consequence, a long

time non-collapsing result was established, which generalizes Perelman's short time result. In [22], under the assumption that Ricci curvature is non-negative and the injectivity is bounded from below, Zhang proved a global upper bound for the fundamental solution of a heat equation introduced by Perelman under backward Ricci flow, i.e.

$$\begin{cases} \partial_t g = -2\text{Ric}, \\ \Delta u - \partial_t u - Ru = 0. \end{cases}$$

Along flow (1.1), we consider the following conjugate heat equation

$$(1.2) \quad \partial_t u(x, t) + \Delta u(x, t) - S(x, t)u(x, t) = 0.$$

In [15, 20], some point-wise gradient estimates for the positive solutions of (1.1) was obtained, which can be viewed as the Li-Yau estimate for the parabolic kernel of the Schrödinger operator in [13, 17, 18].

The main goal of this paper is to establish some certain Sobolev inequalities under system (1.1) and a global upper bound for the fundamental solution of heat equation (1.2) under the extended Ricci flow. Via the monotonicity of the entropies, we obtain the following Sobolev inequality.

Theorem 1.1. *Let $(g(x, t), \phi(x, t))$ be a solution of the system (1.1) for $t \in [0, T_0)$ with initial metric g_0 , where $T_0 \leq \infty$ is the life span of (1.1). Let A_0 and B_0 be positive numbers such that the following L^2 Sobolev inequality holds initially, i.e. for any $v \in W^{1,2}(M, g_0)$,*

$$\left(\int_M v^{\frac{2n}{n-2}} d\mu(g_0) \right)^{\frac{n-2}{n}} \leq A_0 \int_M (|\nabla v|^2 + \frac{1}{4}Sv^2) d\mu(g_0) + B_0 \int_M v^2 d\mu(g_0).$$

Then for all $v \in W^{1,2}(M, g(t))$, we have

$$(1.3) \quad \left(\int_M v^{\frac{2n}{n-2}} d\mu(g(t)) \right)^{\frac{n-2}{n}} \leq A(t) \int_M (|\nabla v|^2 + \frac{1}{4}Sv^2) d\mu(g(t)) + B(t) \int_M v^2 d\mu(g(t)).$$

Where $A(t)$ and $B(t)$ are positive constants depending only on A_0 , $(1+t)B_0$ and n .

By the above Sobolev inequality, combining with Morse's iteration and Davies heat kernel estimate, we prove following Gaussian type upper bound for the fundamental solutions of (1.2) and the bound does not depend on the lower bound of injective radius, but depends on the the first eigenvalue of the entropy, which is different from Zhang's result in [22]. More precisely, we prove

Theorem 1.2. *Let $(g(x, t), \phi(x, t))$ be a smooth solution of system (1.1) in $M \times [0, T]$ and $G(x, t; y, T)$ be a fundamental solution of the following backward conjugate heat equation (1.2), that is*

$$\begin{cases} \Delta_x G(x, t; y, T) + \partial_t G(x, t; y, T) - S(x, t)G(x, t; y, T) = 0, & 0 \leq t < T; \\ G(x, t; y, T) = \delta(x, y), & t = T. \end{cases}$$

Assume further that $Sy \geq 0$ and the first eigenvalue λ_0 of entropy $\inf_{\|v\|_2=1} \int_M (4|\nabla v|^2 + Sv^2) d\mu(g_0)$ is positive. Then for any $t \in (0, T)$, and $x, y \in M$, we have the following estimates

$$(1.4) \quad G(x, t; y, T) \leq \frac{c}{|B(y, \sqrt{T-t}, T)|_T} \exp \frac{-c_1 d^2(x, y, T)}{T-t},$$

where c_1 is a constant depending only on dimension n) and c is a constant depending on dimension n , λ_0 and initial metric g_0 . Here $d(x, y, T)$ denotes the distance between x and y with respect to metric $g(T)$, $B(y, \sqrt{T-t}, T)$ denotes the geodesic ball centered at y with radius $\sqrt{T-t}$, and $|B(y, \sqrt{T-t}, T)|_T$ denotes the volume of the ball $B(y, \sqrt{T-t}, T)$ with respect to metric $g(T)$.

The rest of the paper is organized as follows. The evolution formulas of entropies under system (1.1) are given in section 2. Corresponding Sobolev inequalities along the extended Ricci flow system are derived in section 3. In section 4, we prove Theorem 1.2.

2. ENTROPIES OF THE EXTENDED RICCI FLOW

In this section, we recall the definitions of entropies via corresponding conjugate heat equation just as Perelman has done in Ricci flow in [19]. Through direct computations, we obtain the monotone quantities of the entropies. Although the monotonicity of the entropies has been obtained in [8], for the completeness we will give a different computation to obtain the evolution equations of entropies without using the entropies' invariance under diffeomorphisms.

Let $u(x, t)$ be a positive solution to the conjugate heat equation (1.2)

$$H^*u := \Delta u - Su + \partial_t u = 0.$$

From the equation (1.2) and the evolution equations (1.1), it follows easily that

$$\frac{d}{dt} \int_M u(x, t) d\mu(g(t)) = \int_M (\partial_t - S)u d\mu(g(t)) = \int_M H^*u d\mu(g(t)) = 0.$$

Here we used the fact that M is closed. Therefore, we assume that $u(x, t)$ satisfies

$$(2.1) \quad \int_M u(x, t) d\mu(g(t)) = 1$$

for any $t \in [0, T]$.

Via the positive solution u of (1.2), the entropies are defined (see for example [8]) as follows

Definition 2.1. *F entropy is defined as the following integration*

$$(2.2) \quad F(t) := \int_M \left(Su + \frac{|\nabla u|^2}{u} \right) d\mu(g(t)),$$

and *W entropy is defined by*

$$(2.3) \quad W(t) := \int_M \left[\tau \left(Su + \frac{|\nabla u|^2}{u} \right) - u \ln u - \frac{n}{2} \ln(4\pi\tau)u - nu \right] d\mu(g(t)),$$

where τ is a scaling factor satisfied $\frac{d\tau}{dt} = -1$.

In order to simplify the computations in this paper, we introduce a function $f(x, t)$ given by:

$$u(x, t) = \frac{e^{-f}}{(4\pi\tau)^{\frac{n}{2}}},$$

where $\frac{d\tau}{dt} = -1$. Then it follows that

$$(2.4) \quad f = -\ln u - \frac{n}{2}(\ln 4\pi\tau).$$

With the above preparations, we now give a direct calculation of the following monotonicity formula.

Proposition 2.1. (see also Lemma 3.4 and Theorem 6.1 in [8]) *Let $(g, \phi)(t)$ be a solution of (1.1) and $u(x, t)$ be a positive solution of (1.2). Then*
(I) F entropy is nondecreasing in t . More precisely,

$$(2.5) \quad \frac{d}{dt}F(t) = 2 \int_M (|Sy + \nabla^2 f|^2 + 2|\Delta\phi - d\phi(\nabla f)|^2) u d\mu(g(t)) \geq 0.$$

(II) W entropy is nondecreasing in t . More precisely,

$$(2.6) \quad \frac{d}{dt}W(t) = \int_M \left(2\tau|Sy + \nabla^2 f - \frac{g}{2\tau}|^2 + 4\tau|\Delta\phi - d\phi(\nabla f)|^2 \right) u d\mu(g(t)) \geq 0.$$

Proof. We start by direct calculations that

$$(2.7) \quad H^*(u \ln u) = \frac{|\nabla u|^2}{u} + Su.$$

and

$$(2.8) \quad \begin{aligned} H^*\left(\frac{|\nabla u|^2}{u} + Su\right) &= \frac{2}{u} \left(\frac{u_i u_j}{u} - u_{ij} \right)^2 + \frac{2S_{ij} u_i u_j}{u} + \frac{2R_{ij} u_i u_j}{u} \\ &\quad + 4\langle \nabla u, \nabla S \rangle + 2u\Delta S + 2(|Sy|^2 + 2|\Delta\phi|^2)u, \end{aligned}$$

here we used the evolution equation $\partial_t S = \Delta S + 2|S_{ij}|^2 + 4|\Delta\phi|^2$ (see for example [8, 20]).

Taking time derivative yields

$$\begin{aligned} \frac{d}{dt}F &= \frac{d}{dt} \int_M \left(Su + \frac{|\nabla u|^2}{u} \right) d\mu = \int_M (\partial_t - S) \left(Su + \frac{|\nabla u|^2}{u} \right) d\mu \\ &= \int_M H^* \left(Su + \frac{|\nabla u|^2}{u} \right) d\mu. \end{aligned}$$

Substituting (2.8) to the above equality, we get

$$(2.9) \quad \begin{aligned} \frac{d}{dt}F &= \int_M \left[\frac{2}{u} \left(\frac{u_i u_j}{u} - u_{ij} \right)^2 + \frac{2S_{ij} u_i u_j}{u} + \frac{2R_{ij} u_i u_j}{u} \right. \\ &\quad \left. + 4\langle \nabla u, \nabla S \rangle + 2u\Delta S + 2(|Sy|^2 + 2|\Delta\phi|^2)u \right] d\mu. \end{aligned}$$

By integration by parts and the contracted second Bianchi identity, we have

$$(2.10) \quad \begin{aligned} \int_M \langle \nabla u, \nabla S \rangle d\mu &= \int_M \langle \nabla u, \nabla (R - 2|d\phi|^2) \rangle d\mu = \int_M (2u_i \nabla_j R_{ij} - 4u_i \phi_j \phi_{ij}) d\mu \\ &= \int_M (-2u_{ij} R_{ij} + 4u_{ij} \phi_j \phi_i + 4u_i \phi_i \Delta\phi) d\mu \\ &= \int_M (-2u_{ij} S_{ij} + 4u_i \phi_i \Delta\phi) d\mu. \end{aligned}$$

Substituting (2.10) into (2.9), we obtain

$$\begin{aligned} \frac{d}{dt}F &= \int_M \left[\frac{2}{u} \left(\frac{u_i u_j}{u} - u_{ij} \right)^2 + \frac{2S_{ij} u_i u_j}{u} + \frac{2R_{ij} u_i u_j}{u} \right. \\ &\quad \left. + 2\langle \nabla u, \nabla S \rangle + 2(|Sy|^2 + 2|\Delta\phi|^2)u \right] d\mu \\ &= \int_M \left[\frac{2}{u} \left(\frac{u_i u_j}{u} - u_{ij} \right)^2 + \frac{2S_{ij} u_i u_j}{u} + \frac{2R_{ij} u_i u_j}{u} \right. \\ &\quad \left. - 4u_{ij} S_{ij} + 8u_i \phi_i \Delta\phi + 2(|Sy|^2 + 2|\Delta\phi|^2)u \right] d\mu. \end{aligned}$$

Replacing u with f in the above equality, we deduce

$$\begin{aligned} \frac{d}{dt}F &= \int_M \left[|f_{ij}|^2 + 2S_{ij} f_{ij} + |S_{ij}|^2 + 2|\Delta\phi|^2 + 2|2d\phi(\nabla f)|^2 \right. \\ &\quad \left. - 4\Delta\phi(d\phi(\nabla f)) \right] d\mu \\ &= 2 \int_M \left[|Sy + \nabla^2 f|^2 + 2|\Delta\phi - d\phi(\nabla f)|^2 \right] u d\mu, \end{aligned}$$

which gives formula (2.5).

From the definition of W entropy, it follows that

$$\frac{d}{dt}W(t) = \int_M H^*(\tau(\frac{|\nabla u|^2}{u} + Su)) - H^*(u \ln u) - \frac{n}{2} H^*(u \ln \tau) d\mu.$$

Substituting (2.7) and (2.8) to the above equality, we get

$$\begin{aligned} \frac{d}{dt}W &= \tau \frac{d}{dt}F - 2F + \frac{n}{2\tau} \\ (2.11) \quad &= \tau \frac{d}{dt}F - 2 \int_M (Su + |\nabla f|^2 u) d\mu(g(t)) + \frac{n}{2\tau}. \end{aligned}$$

By the definition of f and integrations by parts, we deduce

$$\int_M (Su + \frac{|\nabla u|^2}{u}) d\mu(g(t)) = \int_M (Su - \nabla u \nabla f) d\mu = \int_M (Su + \Delta f u) d\mu.$$

Substituting the above equality and equality (2.5) into (2.11), we have

$$\begin{aligned} \frac{d}{dt}W &= 2\tau \int_M (|Sy + \nabla^2 f|^2 + 2|\Delta\phi - d\phi(\nabla f)|^2) u d\mu - 2 \int_M (S + \Delta f) u d\mu + \frac{n}{2\tau} \\ &= \int_M 2\tau \left[|Sy + Hess(f) - \frac{g}{2\tau}|^2 + 2|\Delta\phi - d\phi(\nabla f)|^2 \right] u d\mu. \end{aligned}$$

Thus we complete the proof. \square

Similarly as the Ricci flow, one can define a family of generalized W entropy along the extended Ricci flow by

$$\begin{aligned} W(a, t) &:= \int_M \left(\frac{a^2 \tau}{2\pi} (Su + \frac{|\nabla u|^2}{u}) - u \ln u - \frac{n}{2} \ln(4\pi\tau)u - nu \right) d\mu(g(t)) \\ (2.12) \quad &= \int_M \left(\frac{a^2 \tau}{2\pi} (S + |\nabla f|^2) + f - n \right) u d\mu(g(t)). \end{aligned}$$

Here the second equality is due to the relations between u and f given by (2.4). The more applications of generalized entropy can be found in [9]. Using the calculations in [10], we get the following monotonicity formula of generalized W entropy.

Proposition 2.2. *Let M be a closed Riemannian manifold, $(g, \phi)(t)$ be a solution of (1.1) and $u(x, t)$ be a positive solution of (1.2). Then the generalized $W(a, t)$ entropy is nondecreasing in t and the following inequality holds:*

$$\frac{d}{dt}W(a, t) \geq \frac{a^2\tau}{\pi} \int_M \left(|Sy + \text{Hess}(f) - \frac{g}{2\tau}|^2 + 2|\Delta\phi - d\phi(\nabla f)|^2 \right) u d\mu.$$

Since the proof of this proposition is the same as that in Ricci flow case, we omit the details here. One can find details in Theorem 4.1 in [10].

3. SOBOLEV INEQUALITIES UNDER THE EXTENDED RICCI FLOW

In this section, we mainly use the monotonicity of W entropy to derive a uniform Sobolev inequality along the system (1.1) which will be used in the next section.

To prove Theorem 1.1, we need the following lemma first, which shows the relations between the logarithmic Sobolev inequality, the $W^{1,2}$ Sobolev inequality and the so-called ultracontractivity of the heat semigroup of the associated Schrödinger operator. The proof of this lemma is more or less standard. One can consult more details for example in Theorem 4.2.1 in [23].

Lemma 3.1 (See for example in [23]). *Let (M^n, g) be a closed Riemannian manifold ($n \geq 3$). Then the following inequalities are equivalent (up to constants).*

(I) *Sobolev inequality: there exists positive constants A and B such that, for all $v \in W^{1,2}(M)$,*

$$\left(\int_M v^{\frac{2n}{n-2}} d\mu \right)^{\frac{n-2}{n}} \leq A \int_M |\nabla v|^2 d\mu + B \int_M v^2 d\mu;$$

(II) *Log-Sobolev inequality: for all $v \in W^{1,2}(M)$ such that $\|v\|_2 = 1$ and all $\epsilon > 0$,*

$$\int_M v^2 \ln v^2 d\mu \leq \epsilon^2 \int_M |\nabla v|^2 d\mu - \frac{n}{2} \ln \epsilon^2 + BA^{-1} \epsilon^2 + \frac{n}{2} \ln \frac{nA}{2e};$$

(III) *Heat kernel upper bound: for all $t > 0$,*

$$G(x, t; y) \leq \frac{(nA)^{\frac{n}{2}}}{t^{\frac{n}{2}}} e^{A^{-1}Bt};$$

(IV) *Nash inequality: for all $v \in W^{1,2}(M)$,*

$$\|v\|_2^{2+\frac{4}{n}} \leq (A\|\nabla v\|_2^2 + B\|v\|_2^2) \|v\|_1^{\frac{4}{n}}.$$

Due to Lemma 3.1, to prove Theorem 1.1, we only need to prove some log-Sobolev inequalities for any $t \in [0, T_0]$. By the monotonicity of W entropy, we obtain the following log-Sobolev inequality.

Lemma 3.2 (Log-Sobolev Inequality). *Under the assumptions of Theorem 1.1. Then for any $t \in [0, T_0]$, $v \in W^{1,2}(M, g(t))$ with $\int_M v^2 d\mu(g(t)) = 1$ and any $\epsilon > 0$, we have*

$$\begin{aligned} \int_M v^2 \ln v^2 d\mu(g(t)) &\leq \epsilon^2 \int_M (4|\nabla v|^2 + Sv^2) d\mu(g(t)) - n \ln \epsilon \\ &\quad + (t + \epsilon^2) B_0 A_0^{-1} + \frac{n}{2} \ln \left(\frac{nA_0}{2e} \right). \end{aligned} \tag{3.1}$$

Proof. For a fixed $t_0 \in [0, T_0]$ and any $\epsilon > 0$, we set

$$\tau(t) = \epsilon^2 + t_0 - t.$$

Recall that W entropy is defined by

$$W(g, f, t) = \int_M (\tau(S + |\nabla f|^2) + f - n) u d\mu(g(t)).$$

From the monotonicity of the W entropy in Proposition 2.1, we deduce

$$(3.2) \quad \inf_{\int_M u d\mu(g(t_0))=1} W(g(t_0), f, \epsilon^2) \geq \inf_{\int_M u_0 d\mu(g(0))=1} W(g(0), f_0, t_0 + \epsilon^2),$$

one can find a more detailed proof of this property in section 3 of [19]. Here f_0 and f are given by the formulas

$$u_0 = \frac{e^{-f_0}}{(4\pi(t_0 + \epsilon^2))^{\frac{n}{2}}}, \quad u = \frac{e^{-f}}{(4\pi\epsilon^2)^{\frac{n}{2}}}.$$

Using these notations we rewrite (3.2) as

$$\begin{aligned} & \inf_{\int u d\mu(g(t_0))=1} \int_M \left(\epsilon^2 (S + |\nabla \ln u|^2) - \ln u - \frac{n}{2} \ln 4\pi\epsilon^2 \right) u d\mu(g(t_0)) \\ & \geq \inf_{\int u_0 d\mu(g(0))=1} \left(\int_M ((\epsilon^2 + t_0)(S + |\nabla \ln u_0|^2) - \ln u_0 - \frac{n}{2} \ln 4\pi(t_0 + \epsilon^2)) u_0 d\mu(g(0)) \right). \end{aligned}$$

Set $v = \sqrt{u}$ and $v_0 = \sqrt{u_0}$, the above inequality yields

$$\begin{aligned} & \inf_{\int v^2 d\mu(g(t_0))=1} \int_M [\epsilon^2 (S v^2 + 4|\nabla v|^2) - v^2 \ln v^2] d\mu(g(t_0)) - \frac{n}{2} \ln \epsilon^2 \\ & \geq \inf_{\int v_0^2 d\mu(g(0))=1} \int_M ((\epsilon^2 + t_0)(S v_0^2 + 4|\nabla v_0|^2) - v_0^2 \ln v_0^2) d\mu(g(0)) - \frac{n}{2} \ln(t_0 + \epsilon^2). \end{aligned} \quad (3.3)$$

Notice that $\ln x$ is a concave function and $\int v_0^2 d\mu(g(0)) = 1$, thus applying Jensen's inequality we deduce

$$\int_M v_0^2 \ln v_0^{q-2} d\mu(g(0)) \leq \ln \int_M v_0^{q-2} v_0^2 d\mu(g(0)),$$

i.e.

$$\int_M v_0^2 \ln v_0^2 d\mu(g(0)) \leq \frac{n}{2} \ln \|v_0\|_q^2,$$

where $q = \frac{2n}{n-2}$. By the assumption that the Sobolev inequality holds for the initial time $t = 0$, we have

$$\int_M v_0^2 \ln v_0^2 d\mu(g(0)) \leq \frac{n}{2} \ln \left(A_0 \int_M (4|\nabla v_0|^2 + S v_0^2) d\mu(g(0)) + B_0 \right).$$

By the inequality

$$\ln z \leq yz - \ln y - 1,$$

for any $y, z > 0$, we have

$$\int_M v_0^2 \ln v_0^2 d\mu(g(0)) \leq \frac{n}{2} y \left(A_0 \int_M (4|\nabla v_0|^2 + S v_0^2) d\mu(g(0)) + B_0 \right) - \frac{n}{2} \ln y - \frac{n}{2}.$$

In the above inequality we choose $y = 2\frac{t_0 + \epsilon^2}{nA_0}$, we then deduce

$$\begin{aligned} \int_M v_0^2 \ln v_0^2 d\mu(g(0)) &\leq (t_0 + \epsilon^2) \int_M (4|\nabla v_0|^2 + Sv_0^2) d\mu(g(0)) \\ &\quad + \frac{(t_0 + \epsilon^2)B_0}{A_0} - \frac{n}{2} \ln \frac{2(t_0 + \epsilon^2)}{nA_0} - \frac{n}{2}. \end{aligned}$$

Substituting the above inequality to the right-hand side of (3.3), we reach the log-Sobolev inequality:

$$\begin{aligned} \int_M v^2 \ln v^2 d\mu(g(t_0)) &\leq \epsilon^2 \int_M (4|\nabla v|^2 + Sv^2) d\mu(g(t_0)) - n \ln \epsilon \\ &\quad + (t_0 + \epsilon^2)B_0 A_0^{-1} + \frac{n}{2} \ln \left(\frac{nA_0}{2e} \right). \end{aligned}$$

Thus we have completed the proof of the log-Sobolev inequality (3.1), hence Theorem 1.1. \square

Since (M, g_0) is a closed Riemannian manifold, the Sobolev inequality holds as described in section 4.1 in [23], i.e. for any $v \in W^{1,2}(M)$, there exist positive constants A and B depending only on the initial metric g_0 such that

$$\left(\int_M v^{\frac{2n}{n-2}} d\mu(g(0)) \right)^{\frac{n-2}{n}} \leq A \int_M |\nabla v|^2 d\mu(g(0)) + B \int_M v^2 d\mu(g(0)). \quad (3.4)$$

Recall that λ_0 is the first eigenvalue of F entropy as characterized in (2.2), i.e.

$$\lambda_0 = \inf_{\|v\|_2=1} \int_M (4|\nabla v|^2 + Sv^2) d\mu(g(0)). \quad (3.5)$$

This eigenvalue has been studied widely and is a very powerful tool for the understanding of Riemannian manifolds. One can find more details in [9]. If $\lambda_0 > 0$, by Sobolev inequality (3.4), we know that the assumption of Sobolev inequality in Theorem 1.1 at initial time holds with $B_0 = 0$, i.e.

$$\left(\int_M v^{\frac{2n}{n-2}} d\mu(g_0) \right)^{\frac{n-2}{n}} \leq \tilde{A}_0 \int_M (|\nabla v|^2 + \frac{1}{4}Sv^2) d\mu(g_0) \quad (3.6)$$

where \tilde{A}_0 depends only on initial metric g_0 and the first eigenvalue of F entropy λ_0 . Furthermore, the log-Sobolev inequality (3.1) in Lemma 3.2 holds with $B_0 = 0$. Therefore from Theorem 1.1, we have the following corollary:

Corollary 3.1. *Let $(g, \phi)(t)$ be a solution of the system (1.1). Assume $\lambda_0 > 0$. Then there exists a positive number A_0 , depending only on initial metric g_0 and λ_0 , such that for all $v \in W^{1,2}(M, g(t))$, $t \in [0, T_0)$, it holds that*

$$\left(\int_M v^{\frac{2n}{n-2}} d\mu(g(t)) \right)^{\frac{n-2}{n}} \leq A_0 \int_M (|\nabla v|^2 + \frac{1}{4}Sv^2) d\mu(g(t)). \quad (3.7)$$

4. THE PROOF OF THEOREM 1.2

In this section, we establish a certain Gaussian type upper bound for the fundamental solutions of the conjugate heat equation with proper lower bound of Ricci curvature. This Gaussian upper bound in Ricci flow was proved by Zhang in [21] with the assumption of the lower bound of injectivity with the help of Sobolev inequality by Heybey (c.f. in [7]), but here by using the uniform Sobolev inequality in corollary 4.1, we derive the similar Gaussian upper bound without any assumption of the lower bound of injectivity. To prove the Gaussian upper bound, we need the following interpolation theorem first. The proof is standard by using maximum principle (see also [13, 15, 18, 20]).

Theorem 4.1. *Let $(g, \phi)(t)$ be a solution of (1.1) and $u(x, t)$ be a positive solution to heat equation*

$$(4.1) \quad \Delta u - \partial_t u = 0$$

for $t \in [0, T]$. Then it holds that

$$(4.2) \quad \frac{|\nabla u(x, t)|}{u(x, t)} \leq \sqrt{\frac{1}{t}} \sqrt{\ln \frac{A}{u(x, t)}}$$

for $(x, t) \in M \times [0, T]$. Here $A = \sup_{M \times [0, T]} u$.

Moreover, for any $\delta > 0$, $x, y \in M$ and $0 < t < T$, the following interpolation inequality holds

$$(4.3) \quad u(y, t) \leq A^{\frac{\delta}{1+\delta}} u^{\frac{1}{1+\delta}}(x, t) \exp\left(\frac{d^2(x, y, t)}{4t\delta}\right).$$

Proof. Using the heat equation (4.1), we have the following computations:

$$\begin{aligned} (\Delta - \partial_t)(u \ln \frac{A}{u}) &= \Delta u \ln \frac{A}{u} + u \Delta(\ln \frac{A}{u}) + 2\nabla u \nabla \ln \frac{A}{u} - \partial_t u \ln \frac{A}{u} - u \partial_t(\ln \frac{A}{u}) \\ &= \Delta u \ln \frac{A}{u} + u(-\frac{\Delta u}{u} + \frac{|\nabla u|^2}{u^2}) - 2\frac{|\nabla u|^2}{u} - \Delta u \ln \frac{A}{u} + \partial_t u \\ (4.4) \quad &= -\frac{|\nabla u|^2}{u}, \end{aligned}$$

$$\begin{aligned} \Delta(\frac{|\nabla u|^2}{u}) &= \frac{\Delta|\nabla u|^2}{u} + \Delta(\frac{1}{u})|\nabla u|^2 + 2\nabla|\nabla u|^2 \nabla(\frac{1}{u}) \\ (4.5) \quad &= \frac{\Delta|\nabla u|^2}{u} + (\frac{2|\nabla u|^2}{u^3} - \frac{\Delta u}{u^2})|\nabla u|^2 - 4\frac{u_i u_j u_{ij}}{u^2}, \end{aligned}$$

and

$$(4.6) \quad \partial_t(\frac{|\nabla u|^2}{u}) = \frac{\partial_t|\nabla u|^2}{u} - \frac{|\nabla u|^2}{u^2} \partial_t u = \frac{2\langle \nabla u, \nabla \Delta u \rangle + 2S_{ij} u_i u_j}{u} - \frac{|\nabla u|^2}{u^2}.$$

Combining (4.5) and (4.6) together, we derive

$$\begin{aligned} (\Delta - \partial_t)(\frac{|\nabla u|^2}{u}) &= \frac{\Delta|\nabla u|^2}{u} + \frac{2|\nabla u|^4}{u^3} - 4\frac{u_i u_j u_{ij}}{u^2} - \frac{2\langle \nabla u, \nabla \Delta u \rangle + 2S_{ij} u_i u_j}{u} \\ &= \frac{2u_{ij}^2 + 4u_i u_j \phi_i \phi_j}{u} + \frac{2|\nabla u|^4}{u^3} - 4\frac{u_i u_j u_{ij}}{u^2} \\ (4.7) \quad &= \frac{4}{u} |d\phi(\nabla u)|^2 + \frac{2}{u} |u_{ij} - \frac{u_i u_j}{u}|^2. \end{aligned}$$

Combining (4.4) and (4.7), we have

$$\begin{aligned}
 (\Delta - \partial_t)\left(\frac{t|\nabla u|^2}{u} - u \ln \frac{A}{u}\right) &= -\frac{|\nabla u|^2}{u} + \frac{4}{u}|d\phi(\nabla u)|^2 + \frac{2}{u}\left|u_{ij} - \frac{u_i u_j}{u}\right|^2 + \frac{|\nabla u|^2}{u} \\
 &= \frac{4}{u}|d\phi(\nabla u)|^2 + \frac{2}{u}\left|u_{ij} - \frac{u_i u_j}{u}\right|^2 \\
 (4.8) \qquad \qquad \qquad &\geq 0.
 \end{aligned}$$

Notice that $A = \sup_{M \times [0, T]} u$, then at $t = 0$ we have

$$\frac{t|\nabla u|^2}{u} - u \ln \frac{A}{u} = -u \ln \frac{A}{u} \leq 0.$$

From (4.8), the maximum principle implies that

$$\frac{t|\nabla u|^2}{u} - u \ln \frac{A}{u} \leq 0,$$

which gives (4.2).

Set $\ell(x, t) = \ln \frac{A}{u(x, t)}$, then inequality (4.2) yields

$$|\nabla \sqrt{\ell(x, t)}| = \frac{1}{2} \left| \frac{\nabla u}{u \sqrt{\ell}} \right| \leq \frac{1}{\sqrt{4t}}.$$

Thus, for any $x, y \in M$, we can integrate the above inequality along a minimizing geodesic joining x and y to get

$$\sqrt{\ln \frac{A}{u(x, t)}} \leq \sqrt{\ln \frac{A}{u(y, t)}} + \frac{d(x, y, t)}{\sqrt{4t}},$$

Thus, for any $\delta > 0$ we have

$$\begin{aligned}
 \ln \frac{A}{u(x, t)} &\leq \ln \frac{A}{u(y, t)} + \frac{d^2(x, y, t)}{4t} + \sqrt{\ln \frac{A}{u(y, t)}} \frac{d(x, y, t)}{\sqrt{t}} \\
 &\leq \ln \frac{A}{u(y, t)} + \frac{d^2(x, y, t)}{4t} + \delta \ln \frac{A}{u(y, t)} + \frac{d^2(x, y, t)}{4t\delta},
 \end{aligned}$$

which immediately gives (4.3). \square

Now we turn to prove Theorem 1.2. With the help of the uniform Sobolev inequality in Corollary 3.1 and the above interpolation theorem, we will begin with the traditional method of establishing a mean value inequality via Moser's iteration and a weighted estimate in the spirit of Davies in [3] to give the full proof of the Gaussian type upper bound (see also [12, 22]).

Proof of Theorem 1.2. We divide the proof into two steps.

Step 1. By using Morse's iteration, we prove a mean value inequality for the positive solution u of conjugate equation (1.2).

For any arbitrary constant $p \geq 1$, it follows trivially that

$$(4.9) \qquad \qquad \qquad \Delta u^p - p S u^p + \partial_t u^p \geq 0.$$

Define the region

$$Q_{\sigma r} := \{(y, s) \mid y \in M, t \leq s \leq t + (\sigma r)^2, d(x, y, s) \leq \sigma r\},$$

with $r > 0$, $1 < \sigma \leq 2$.

Let $\varphi(\rho) : [0, +\infty) \rightarrow [0, 1]$ be a smooth function satisfying: $|\varphi'| \leq \frac{2}{(\sigma-1)r}$, $\varphi' \leq 0$, $\varphi \geq 0$, $\varphi(\rho) = 1$ when $0 \leq \rho \leq r$, and $\varphi(\rho) = 0$ when $\rho \geq \sigma r$. Let $\eta(s) : [0, +\infty) \rightarrow [0, 1]$ be a smooth function satisfying: $|\eta'| \leq \frac{2}{(\sigma-1)^2 r^2}$, $\eta' \leq 0$, $\eta \geq 0$, $\eta(s) = 1$ when $s \leq t + r^2$, and $\eta(s) = 0$ when $t + (\sigma r)^2 \leq s \leq T$. Define a cut-off function $\psi(y, s)$ by

$$\psi(y, s) = \varphi(d(y, x, s))\eta(s).$$

Writing $\omega = u^p$, multiplying $\omega\psi^2$ to (4.9) for $p \geq 1$ and integrating by parts yields

$$\int_{Q_{\sigma r}} \nabla(\omega\psi^2) \nabla \omega dg(y, s) ds + p \int_{Q_{\sigma r}} S \omega^2 \psi^2 dg(y, s) ds \leq \int_{Q_{\sigma r}} (\partial_s \omega) \omega \psi^2 dg(y, s) ds. \quad (4.10)$$

Integrating by parts, the right hand side of (4.10) yields

$$\begin{aligned} \int_{Q_{\sigma r}} (\partial_s \omega) \omega \psi^2 dg(y, s) ds &= - \int_{Q_{\sigma r}} \omega^2 \psi \partial_s \psi dg(y, s) ds + \frac{1}{2} \int_{Q_{\sigma r}} (\psi \omega)^2 S dg(y, s) ds \\ &\quad - \frac{1}{2} \int_{B_{\sigma r}(t)} (\psi \omega)^2 dg(y, t). \end{aligned}$$

By the non-negativity of Sy and using the identity (c.f. [2, 8])

$$\partial_s d(x, y, s) = - \int_0^{d(x, y, s)} Sy(\gamma'(\tau), \gamma'(\tau)) d\tau \leq 0,$$

we have

$$\partial_s \psi = \eta(s) \varphi'(d(y, x, s)) \partial_s d(x, y, s) + \varphi(d(y, x, s)) \eta'(s) \geq \varphi(d(y, x, s)) \eta'(s).$$

Hence

$$\begin{aligned} \int_{Q_{\sigma r}} (\partial_s \omega) \omega \psi^2 dg(y, s) ds &\leq \int_{Q_{\sigma r}} \omega^2 \psi \varphi(d(y, x, s)) |\eta'(s)| dg(y, s) ds \\ &\quad + \frac{1}{2} \int_{Q_{\sigma r}} (\psi \omega)^2 S dg(y, s) ds - \frac{1}{2} \int_{B_{\sigma r}(t)} (\psi \omega)^2 dg(y, t). \end{aligned} \quad (4.11)$$

By direct calculation, we get

$$\int_{Q_{\sigma r}} \nabla(\omega\psi^2) \nabla \omega dg(y, s) ds = \int_{Q_{\sigma r}} |\nabla(\omega\psi)|^2 dg(y, s) ds - \int_{Q_{\sigma r}} |\nabla \psi|^2 \omega^2 dg(y, s) ds. \quad (4.12)$$

From (4.10), (4.11), (4.12), and the assumptions of $p \geq 1$ and $S \geq 0$, we deduce

$$\begin{aligned} &\int_{Q_{\sigma r}} |\nabla(\omega\psi)|^2 dg(y, s) ds + \frac{1}{2} \int_{Q_{\sigma r}} S(\omega\psi)^2 + \frac{1}{2} \int_{B_{\sigma r}(t)} (\psi \omega)^2 dg(y, t) \\ &\leq \int_{Q_{\sigma r}} \omega^2 \psi \varphi(d(y, x, s)) |\eta'(s)| dg(y, s) ds + \int_{Q_{\sigma r}} |\nabla \psi|^2 \omega^2 dg(y, s) ds \\ &\leq \frac{c}{(\sigma-1)^2 r^2} \int_{Q_{\sigma r}} \omega^2 dg(y, t). \end{aligned} \quad (4.13)$$

Using Hölder inequality, we get

$$\int (\psi \omega)^{2(1+\frac{2}{n})} dg \leq \left(\int (\psi \omega)^{\frac{2n}{n-2}} dg \right)^{\frac{n-2}{n}} \left(\int (\psi \omega)^2 dg \right)^{\frac{2}{n}}. \quad (4.14)$$

From Corollary 3.1, we know that for any $t \in (0, T)$, the following Sobolev imbedding inequality holds:

$$(4.15) \quad \left(\int (\psi\omega)^{\frac{2n}{n-2}} dg(s) \right)^{\frac{n-2}{n}} \leq A_0 \int (|\nabla(\psi\omega)|^2 + S(\psi\omega)^2) dg(s),$$

where A_0 depends only on dimension n , λ_0 and initial metric g_0 .

Substituting (4.15) to (4.14), we get

$$\int_{B_{\sigma r}(s)} (\psi\omega)^{2(1+\frac{2}{n})} dg(s) \leq A_0 \int_{B_{\sigma r}(s)} (|\nabla(\psi\omega)|^2 + S(\psi\omega)^2) dg \left(\int_{B_{\sigma r}(s)} (\psi\omega)^2 dg \right)^{\frac{2}{n}}$$

Setting $\theta = 1 + \frac{2}{n}$, integrating the above inequality with respect to s on $[t, t + (\sigma r)^2]$ and using (4.13), we reach

$$\int_{Q_{\sigma r}} (\psi\omega)^{2\theta} dg(y, s) ds \leq A_0 \left(\frac{1}{(\sigma - 1)^2 r^2} \int_{Q_{\sigma r}(x, t)} \omega^2 dg(y, s) ds \right)^\theta.$$

By the definition of ψ , we conclude

$$(4.16) \quad \int_{Q_r} \omega^{2\theta} dg(y, s) ds \leq A_0 \left(\frac{1}{(\sigma - 1)^2 r^2} \int_{Q_{\sigma r}(x, t)} \omega^2 dg(y, s) ds \right)^\theta.$$

Now we choose the sequences of σ_i and p_i as

$$\sigma_0 = 2, \sigma_i = 2 - \sum_{j=1}^i 2^{-j}, p_i = \theta^i,$$

then inequality (4.16) gives that

$$\|u^2\|_{L^{\theta^{i+1}}(\sigma_{i+1}r)} \leq A_0^{(\frac{1}{\theta})^{i+1}} \left(\frac{\sigma_{i+1}^2}{(\sigma_i - \sigma_{i+1})^2 r^2} \right)^{(\frac{1}{\theta})^i} \|u^2\|_{L^{\theta^i}(\sigma_i r)},$$

which gives a L^2 mean value inequality

$$(4.17) \quad \sup_{Q_{r/2}(x, t)} u^2 \leq \frac{c}{r^{2+n}} \int_{Q_r(x, t)} u^2 dg(y, s) ds.$$

Here constant c depends on dimension n , λ_0 and initial metric $g(0)$. From here, by a generic trick of Li and Schoen (c.f. in [12, 14]), we arrive at a L^1 mean value inequality: for $r > 0$,

$$(4.18) \quad \sup_{Q_{r/2}(x, t)} u \leq \frac{c}{r^{2+n}} \int_{Q_r(x, t)} u dg(y, s) ds.$$

For $y \in M$ and $s > t$, applying (4.18) on $u = G(\cdot, \cdot : y, T)$ with $r = \sqrt{\frac{T-t}{2}}$ and using the fact $\int_M u(z, \tau) dg(z, \tau) d\tau = 1$, we conclude

$$(4.19) \quad G(x, t; y, T) \leq \frac{c}{(T-t)^{\frac{n}{2}}}.$$

Step 2. Using methods of the exponential weight due to Davies in [3] to prove the full bound with the exponential term.

It is clear that we only have to deal with the case $d(x_0, y_0, T) \geq 2\sqrt{T-t}$. Otherwise, by (4.19), the Gaussian type upper bound (1.4) holds obviously. Pick a point $x_0 \in M$, a

number $\lambda < 0$ which is determined later and a function $f \in L^2(M, g(T))$. Consider the functions $u(x, t)$ and $F(x, t)$ defined by

$$\begin{aligned} u(x, t) &=: \int_M G(x, t; y, T) e^{-\lambda d(y, x_0, T)} f(y) dg(y, T) \\ F(x, t) &=: e^{\lambda d(x, x_0, t)} u(x, t). \end{aligned}$$

It is clear that u is a solution of (1.2) with initial data

$$u(x, T) = e^{-\lambda d(x, x_0, T)} f(x).$$

Direct calculation shows

$$\begin{aligned} \partial_t \int F^2(x, t) dg(x, t) &= \partial_t \int e^{2\lambda d(x, x_0, t)} u^2(x, t) dg(x, t) \\ &= 2\lambda \int e^{2\lambda d(x, x_0, t)} \partial_t d(x, x_0, t) u^2(x, t) dg(x, t) \\ &\quad - \int e^{2\lambda d(x, x_0, t)} u^2(x, t) S(x, t) dg(x, t) \\ &\quad - 2 \int e^{2\lambda d(x, x_0, t)} u(x, t) [\Delta u - Su] dg(x, t) \\ &\geq -2 \int e^{2\lambda d(x, x_0, t)} u(x, t) \Delta u dg(x, t), \end{aligned}$$

where the last inequality has used the assumption that $Sy \geq 0$, $\lambda < 0$, and $\partial_t d(x, x_0, t) \leq 0$. Using integration by parts, we turn this inequality into

$$\begin{aligned} \partial_t \int F^2(x, t) dg(x, t) &\geq 4\lambda \int e^{2\lambda d(x, x_0, t)} u(x, t) \nabla u \nabla d(x, x_0, t) dg(x, t) \\ &\quad + 2 \int e^{2\lambda d(x, x_0, t)} |\nabla u|^2 dg(x, t). \end{aligned}$$

Direct calculation gives

$$\begin{aligned} \int |\nabla F(x, t)|^2 dg(x, t) &= \int |\nabla(u(x, t) e^{\lambda d(x, x_0, t)})|^2 dg(x, t) \\ &= \int e^{2\lambda d(x, x_0, t)} |\nabla u|^2 dg(x, t) \\ &\quad + 2\lambda \int e^{2\lambda d(x, x_0, t)} u(x, t) \nabla u \nabla d(x, x_0, t) dg(x, t) \\ &\quad + \lambda^2 \int e^{2\lambda d(x, x_0, t)} |\nabla d|^2 u^2 dg(x, t). \end{aligned}$$

Combining the last two expressions, we deduce

$$\partial_t \int F^2(x, t) dg(x, t) \geq 2 \int |\nabla F(x, t)|^2 dg(x, t) - 2\lambda^2 \int e^{2\lambda d(x, x_0, t)} u^2 dg(x, t),$$

which shows

$$\partial_t \int F^2(x, t) dg(x, t) \geq -2\lambda^2 \int F^2(x, t) dg(x, t).$$

Integrating on $[t, T]$, we arrive at the following L^2 estimate

$$\int F^2(x, t) dg(x, t) \leq e^{2\lambda^2(T-t)} \int F^2(x, T) dg(x, T) = e^{2\lambda^2(T-t)} \int f^2(x) dg(x, T). \quad (4.20)$$

Therefore, by the mean value inequality (4.17) with $r = \sqrt{\frac{T-t}{2}}$, the following holds

$$\begin{aligned} u^2(x, t) &\leq \frac{c}{(T-t)^{1+\frac{n}{2}}} \int_t^{\frac{T+t}{2}} \int_{B(x, \sqrt{\frac{T-t}{2}}, \tau)} u^2(z, \tau) dg(z, \tau) d\tau \\ &\leq \frac{c}{(T-t)^{1+\frac{n}{2}}} \int_t^{\frac{T+t}{2}} \int_{B(x, \sqrt{\frac{T-t}{2}}, \tau)} e^{-2\lambda d(z, x_0, \tau)} F^2(z, \tau) dg(z, \tau) d\tau. \end{aligned}$$

In particular, this holds at $x = x_0$. Therefore, by the assumption that $\lambda < 0$, we get

$$u^2(x_0, t) \leq \frac{ce^{-2\lambda\sqrt{\frac{T-t}{2}}}}{(T-t)^{1+\frac{n}{2}}} \int_t^{\frac{T+t}{2}} \int_{B(x_0, \sqrt{\frac{T-t}{2}}, \tau)} F^2(z, \tau) dg(z, \tau) d\tau.$$

From (4.20), it follows that

$$u^2(x_0, t) \leq \frac{ce^{2\lambda^2(T-t)-2\lambda\sqrt{\frac{T-t}{2}}}}{(T-t)^{\frac{n}{2}}} \int f^2(y) dg(y, T).$$

i.e.

$$\left(\int G(x_0, t; z, T) e^{-\lambda d(z, x_0, T)} f(z) dg(z, T) \right)^2 \leq \frac{ce^{2\lambda^2(T-t)-2\lambda\sqrt{\frac{T-t}{2}}}}{(T-t)^{\frac{n}{2}}} \int f^2(y) dg(y, T). \quad (4.21)$$

Now we fix y_0 such that $d(y_0, x_0, T)^2 \geq 4(T-t)$. Then it is clear that, by $\lambda < 0$ and the triangle inequality,

$$-\lambda d(z, x_0, T) \geq -\frac{\lambda}{2} d(x_0, y_0, T)$$

provided by $d(z, y_0, T) \leq \sqrt{T-t}$. Then (4.21) implies

$$\left(\int_{B(y_0, \sqrt{T-t}, T)} G(x_0, t; z, T) f(z) dg(z, T) \right)^2 \leq \frac{ce^{\lambda d(x_0, y_0, T)+2\lambda^2(T-t)-2\lambda\sqrt{\frac{T-t}{2}}}}{(T-t)^{\frac{n}{2}}} \int f^2(y) dg(y, T).$$

From the Cauchy-Schwarz inequality, it follows trivially that

$$2\lambda^2(T-t) - 2\lambda\sqrt{\frac{T-t}{2}} \leq 3\lambda^2(T-t) + \frac{1}{2}.$$

If we choose $\lambda = -\frac{d(x_0, y_0, T)}{b(T-t)}$, then we have

$$\left(\int_{B(y_0, \sqrt{T-t}, T)} G(x_0, t; z, T) f(z) dg(z, T) \right)^2 \leq \frac{ce^{-\frac{c_1 d^2(x_0, y_0, T)}{T-t}}}{(T-t)^{\frac{n}{2}}} \int f^2(y) dg(y, T)$$

with $b > 0$ sufficiently large, and c_1 is an absolute constant. Then by arbitrariness of f , we derive

$$\int_{B(y_0, \sqrt{T-t}, T)} G^2(x_0, t; z, T) dg(z, T) \leq \frac{ce^{-c_1 \frac{d^2(x_0, y_0, T)}{T-t}}}{(T-t)^{\frac{n}{2}}}.$$

Hence, there exists $z_0 \in B(y_0, \sqrt{T-t}, T)$ such that

$$G^2(x_0, t; z_0, T) \leq \frac{ce^{-c_1 \frac{d^2(x_0, y_0, T)}{T-t}}}{(T-t)^{\frac{n}{2}} |B(y_0, \sqrt{T-t}, T)|_T}. \quad (4.22)$$

Finally, let us remind that in [1] the adjoint property of the $G(x_0, t; \cdot, \cdot)$ is obtained, thus we have

$$\Delta_z G(x, t; z, \tau) - \partial_\tau G(x, t; z, \tau) = 0$$

along the extend Ricci flow (1.1). Choosing $\delta = 1$, it then follows from Theorem 4.1 that

$$(4.23) \quad G(x_0, t; y_0, T) \leq \sqrt{G(x_0, t; z_0, T)} \sqrt{A} e^{\frac{d^2(y_0, z_0, T)}{4(T-t)}} \leq e^{1/4} \sqrt{G(x_0, t; z_0, T)} \sqrt{A},$$

where $A = \sup_{M \times [\frac{t+T}{2}, T]} G(x_0, t; \cdot, \cdot)$ and we used $z_0 \in B(y_0, \sqrt{T-t}, T)$. Since (4.19) implies $A \leq \frac{c}{(T-t)^{\frac{n}{2}}}$, then (4.22) and (4.23) immediately yields

$$G(x_0, t; y_0, T)^2 \leq \frac{c}{(T-t)^{\frac{n}{2}}} \frac{1}{(T-t)^{\frac{n}{4}} \sqrt{|B(y_0, \sqrt{T-t}, T)|_T}} e^{-\frac{c_1 d^2(x_0, y_0, T)}{T-t}}.$$

Therefore, by Cauchy-Schwarz inequality, we get

$$\begin{aligned} G(x_0, t; y_0, T) &\leq c \left(\frac{1}{(T-t)^{\frac{n}{2}}} + \frac{1}{|B(y_0, \sqrt{T-t}, T)|_T} \right) e^{-\frac{c_1 d^2(x_0, y_0, T)}{T-t}} \\ &\leq \frac{c}{|B(y_0, \sqrt{T-t}, T)|_T} e^{-\frac{c_1 d^2(x_0, y_0, T)}{T-t}}. \end{aligned}$$

Here c depends on dimension n , λ_0 and initial metric $g(0)$ and c_1 depends only on dimension n . In last inequality, we used volume comparison theorem with the non-negative Ricci curvature. Since x_0 and y_0 are arbitrary, we complete the proof. \square

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